Advanced Homework 1 - Polynomial Approximations

Due: April 14th

This is the first in a series of 5 "advanced" homework assignments that explore taylor series, the number e, imaginary numbers, and finally euler's equation for e^{ix}. These ideas are some of the most beautiful and profound ones in all of mathematics; they run through and tie together everything else we do in calculus.

These assignments test a theory I have: that while we cordone some topics off as "too advanced", they're actually not that hard to start to learn with a gentle introduction; and they're worth the effort. More practically, everyone comes to a class from a different starting place, and with a different learning style. These assignments give you an option to customize your experience: in exchange for doing a little more work you'll learn more (and maybe in way that works better for you), and you'll have more control over your schedule and pacing for the quarter.

Instructions

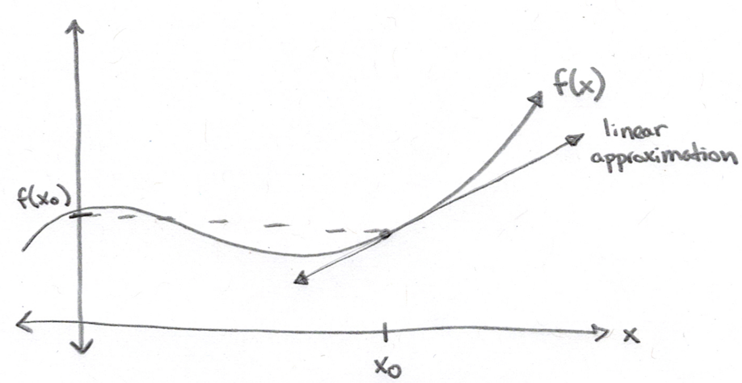
Complete all the problems by hand (or print them) and turn them in stapled, with your name clearly written on the front any time on or before the due date.

You may ask for help on Piazza or during Brendan's office hours, and after you turn it in either your assignment will receive full credit or you'll be asked to correct anything you got wrong to earn full credit.

Polynomial Approximations

If we want approximate some function near some known point x_0, we can use:

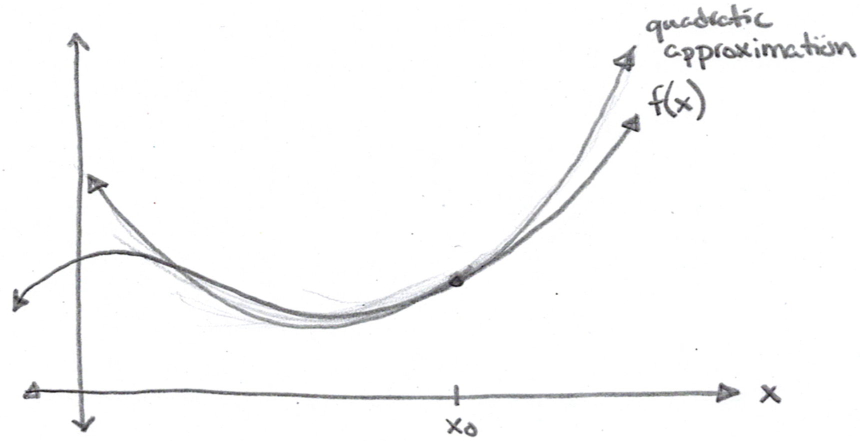
$$f(x) \approx f(x_0) + (x - x_0)f'(x_0)$$



This is called a linear approximation: at one point (x_0), the original function and the approximation have the same value ($f(x_0)$) and the same slope ($f'(x_0)$). And close to that point, they have close to the same value.

Linear approximations are pretty good! But we can do even better with a quadratic approximation.

$$f(x) \approx f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0)$$



Now our approximation has the same value, first derivative, and second derivative at the point x_0 (see problem 3).

And we can add $\frac{1}{2*3}(x - x_0)^3 f'''(x_0)$ for a cubic (3rd degree) approximation, $\frac{1}{2*3*4}(x - x_0)^4 f''''(x_0)$ for a 4th degree approximation, and so on.

Problems

1. Find the linear approximation to the function $f(x) = \sqrt(x)$ near $x_0 = 1$.
2. Use your linear approximation to approximate $sqrt(1.001)$ (the actual value is ~1.00049987506). Were you pretty close?
3. Show that both sides of $f(x) \approx f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0)$ have the same second derivative at the point x_0.
4. Show why the cubic term should have $\frac{1}{2*3}$
5. Show or explain in one sentence why the nth term should be:

$$\frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n$$

(i.e. why is the coefficient $\frac{1}{n!}$?)