Advanced Homework 2 - Taylor Series

Due: April 21st

Taylor Series

In the previous assignment we approximated a function $f(x)$ first with a line:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

Then with a quadratic polynomial:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2$$

and so on. And as we went we saw that in general each term has the form:

$$\frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n$$

This lets us can write the Nth degree polynomial approximation as:

$$\sum_{n=0}^{N} \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n$$

As we add more and more terms, the approximation gets better and better. So we might ask, what happens as we add infinitely many? Then the approximation gets infinitely good; which means the function *equals* the approximation.[1]

$$f(x) = \lim_{N \to \infty} \sum_{n=0}^{N} \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n$$

For example, we can find the Taylor series for $\frac{1}{1+x}$ around $x_0 = 0$:

The 0th term:

$$f(x_0) = \frac{1}{1 + x_0} = \frac{1}{1 + 0} = 1$$

The first term:

$$f'(x) = \frac{1}{(1 + x)^2} * -1$$

[1] This is only true under certain conditions - but for now we'll stick to the conditions where it is true.

$$f'(x_0 = 0) = \frac{-1}{(1+x_0)^2} = \frac{-1}{(1+0)^2} = -1$$

so the first term is:

$$f'(x_0)(x - x_0) = -1(x - 0) = -x$$

The second term:

$$f''(x) = \frac{-1 * -2}{(1+x)^3}$$
$$f''(x_0 = 0) = \frac{-1 * -2}{(1+0)^3} = 2$$

so the second term is:

$$\frac{2}{2}x^2 = x^2$$

And as we continue we find that the derivatives at $x_0 = 0$ cancel out with the $\frac{1}{n!}$ terms but alternate in sign so we have:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + ... = \sum_{n_0}^{\infty} (-1)^n x^n$$

This is kind of a weird point - that a function can be equal to an infinitely long sum of terms based on its derivatives. But you'll see that it can be super useful and let us understand some deep results that would be otherwise pretty intractable.

Problems

1. Find the Taylor series for the following functions using $x_0 = 0$:
   a. $\frac{1}{1-x}$
   b. e^x
   c. $\sin(x)$
   d. $\cos(x)$
2. Use the Taylor series for e^x to show that $\frac{d}{dx}e^x = e^x$ by differentiating each term.
3. Use the Taylor series for $\sin(x)$ and $\cos(x)$ to show that $\frac{d}{dx}\sin(x) = \cos(x)$ and $\frac{d}{dx}\cos(x) = -\sin(x)$.
4. Use the Taylor series for $\frac{1}{1-x}$ to find the sums of the following infinite series:
   a.
   $$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$
   b.
   $$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots$$