Advanced Homework 3 - Euler's Number

Due: May 5th

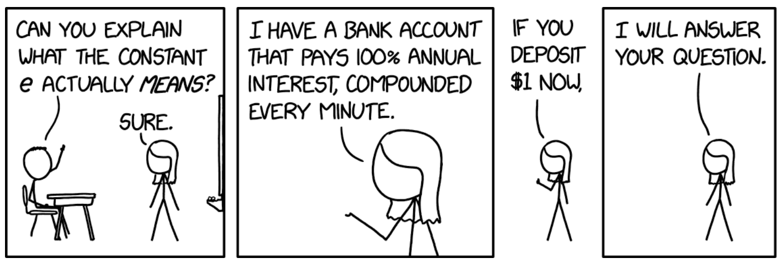

Figure 1: xkcd - "Definition of e"

You'll often see the number e introduced as

$$\lim_{n\to\infty}(1+\frac{1}{n})^n$$

an expression that comes up in compound interest (and is interesting in its own right). Here we'll take a different approach by focusing on its role in the expontential function, but check out the recommendations section at the end for more.

Exponents & Logs

We usually see e as part of the function e^x, or its inverse $\ln(x) = \log_e(x)$. But why is that so common in calculus? Exponential functions are obviously useful, but when we see them elsewhere they're usually use base 10 or base 2.

For example:

- The Richter scale for earthquakes is base 10 (a magnitude 5 earthquake is $10^5/10^1$ = 10,000 times more powerful than a magnitude 1 earthquake)
- The decibel system for sound is base 10 (a 30 decibel sound is 10^3 = 1,000 times louder than a decibel 0 sound).
- Octaves in music are base 2 - moving up an octave *double* the frequency of a tone.
- Bits measure information is base 2

and the list goes on.

Have you ever wondered why when you get to calculus you never see 10^x or 2^x?

Everything is e^x and $\ln(x) = \log_e(x)$. In fact, we really only know how to take $\frac{d}{dx}e^x$ and $\frac{d}{dx}\ln(x)$. Do you even know the derivative of $\frac{d}{dx}(2^x)$?

Derivatives

Let's find out.

Think back to when you first learned derivatives. You learned this definition (or something similar):

$$\frac{d}{dx}f = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

The derivative of f at some point x is the *slope* of f at x And the slope is "rise over run" (which is pretty much where the notation dy/dx comes from). So we compute the *rise* of f between x and another point $x + h$ (rise $= f(x+h) - f(x)$) and put it over the *run* (run $= (x + h) - x = h$). Then we take the limit as we bring $x + h$ closer and closer to x (i.e. $\lim_{h\to 0}$).

The rest of this won't make much sense until you play around with that definition for a little bit, so stop and work through these problems.

You can also reread **chapter 4.1 in the textbook** for a refresher.

Problems 1/3

Use the definition of the derivative to show that:

1. $$\frac{d}{dx}x^2 = 2x$$
2. $$\frac{d}{dx}3x + 2 = 3$$
3. $$\frac{d}{dx}x^n = nx^{n-1}$$

Derivative of the exponential

Ok now let's see if we can figure out $\frac{d}{dx}2^x$. We know $\frac{d}{dx}e^x$ is exactly just e^x. Maybe 2^x is similar?

$$\begin{aligned}
\frac{d}{dx}2^x &= \lim_{h\to 0}\frac{2^{x+h} - 2^x}{h} \\
&= \lim_{h\to 0}\frac{2^x * 2^h - 2^x}{h} \\
&= \lim_{h\to 0} 2^x\left(\frac{2^h - 1}{h}\right) \\
&= 2^x\left(\lim_{h\to 0}\frac{2^h - 1}{h}\right)
\end{aligned}$$

The derivative of 2^x is 2^x times some factor that doesn't depend on x. If we run through the same process for 10^x we find that

$$\frac{d}{dx}10^x = 10^x \left(\lim_{h\to 0} \frac{10^h - 1}{h}\right)$$

And we can go ahead and do it in general for any base b:

$$\frac{d}{dx}b^x = b^x \left(\lim_{h\to 0} \frac{b^h - 1}{h}\right)$$

Problems 2/3

4. Follow the steps we took above to verify that

$$\frac{d}{dx}10^x = 10^x \left(\lim_{h\to 0} \frac{10^h - 1}{h}\right)$$

5. Follow the steps we took above to verify that

$$\frac{d}{dx}b^x = b^x \left(\lim_{h\to 0} \frac{b^h - 1}{h}\right)$$

Finding the perfect base

Now either:

- watch the 3Blue1Brown video on Youtube called **"What's so special about Euler's number e?"** (https://youtu.be/m2MIpDrF7Es) through the section "Deriving the key proportionality property", or
- read **chapter 4.9 in the textbook**.

Both walk through the steps of differentiating b^x and then exploring the leftover term $\lim_{h\to 0} \frac{b^h-1}{h}$.

To summarize:

1. The derivative of the exponential function is proportional to itself: i.e. it's itself times times a constant (the leftover term).
2. That constant is less than 1 for $b < 2$ and greater than 1 for $b > 3$
3. This suggests some *special* number (e) between 2 and 3 that acts as the perfect base to the exponential function so that $\frac{d}{dx}e^x = e^x$.

Both the textbook and the video just jump to the answer, but we have the tools to figure out the number ourselves.

Problems 3/3

6. Find the Taylor series for a function $f(x)$ which is its own derivative (i.e. find a series that's the same after you differentiate it). We did this in the previous homework, but if you can, try to work it out again!

This function is exactly e^x.

7. We know that $b^1 = b$ for any b. So to find the value of e, plug $x = 1$ into your series above – $f(1) = e^1 = e$. You should find something that starts with $1 + 1 + 1/2 + ...$ - what are the rest of the terms? (Write your answer either as a closed form sum, or with ... but showing the general *nth* term).

This is probably the purest explanation of where e comes from, and why we use it everywhere in math. And you'll see in the next two problems why you don't really need any other exponential functions.

8. Use the fact that $2 = e^{\ln(2)}$ to find $\frac{d}{dx}2^x$.
9. Use the fact that $b = e^{\ln(b)}$ to find $\frac{d}{dx}b^x$.

Other recommendations

If you want to see the more classic derivation of e that comes from compounding interest, watch the Numberphile Youtube video called *"e (Euler's Number)"*.

(https://www.youtube.com/watch?v=AuA2EAgAegE)

This is about the definition of e as

$$\lim_{n\to\infty}(1 + 1/n)^n$$

and you'll see why this is the same as what we just found (though I find the final bit a lot less satisfying than this way of doing it).

The Wikipedia page for *"e (mathematical constant)"* is also interesting, if a little overwhelming!