Advanced Homework 4 - Imaginary Numbers

Suggested Due Date: May 22nd

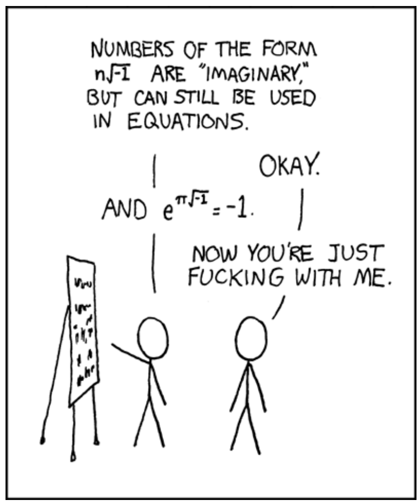

Figure 1: xkcd - "e to the pi times i"

This assignment introduces "imaginary" numbers like $\sqrt{-7}$, and "complex" numbers like $3 + 2\sqrt{-1}$. It will cover:

- the algebra of complex numbers, and how to think about them,
- the connection between complex numbers and rotation, and finally
- euler's equation $e^{ix} = \cos(x) + i\sin(x)$.

To start, watch the video by *Veritasium* on Youtube **"How Imaginary Numbers Were Invented"** (https://youtu.be/cUzklzVXJwo)

Imaginary numbers

Let's start with some definitions.

$\sqrt{-1}$ is i, and for any b, $b\sqrt{-1} = bi$ is called "imaginary". If you have a number like $\sqrt{-9}$ or $\sqrt{-7}$ you want to separate out the i – so the first would be $\sqrt{-9} = \sqrt{9}\sqrt{-1} = 3i$ and the second we would just write $\sqrt{7}i$.

Complex numbers are the sums of imaginary and non-imaginary ("real") numbers, often written in the form $a+bi$ - where a is the real component and b is the imaginary component.

The next step is to take this notion of "components" even futher and think of complex numbers as numbers in a two-dimensional plane. The real components lie along the regular number line with positive numbers to the right and negative numbers, to the left, but now the imaginary components extend this picture into two dimensions along a perpendicular axis.

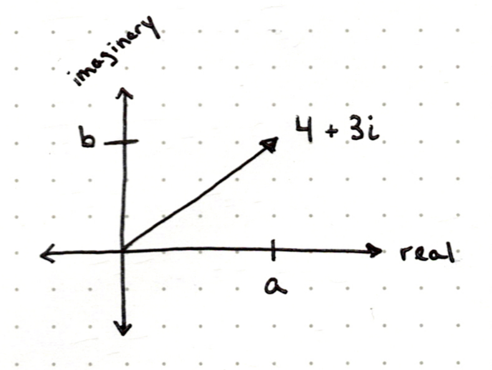

Figure 2: This is the number 4 + 3i in the complex plane as a vector. The x-direction component is a (so 4) and the y direction component is b (3). A complex number with negative b is also possible, and then the vector would be pointing down.

Algebra of complex numbers works about the exact same as regular algebra; you just have to occasionally recall that $i = \sqrt{-1}$ so $i^2 = \sqrt{-1}^2 = -1$. You can always simplify a complex expression into $a + bi$ form.

For example:

$$\begin{aligned}(1+2i)*(2-i) &= 1(2-i)+2i(2-i)\\ &= 2-i+4i-2i^2\\ &= 2+3i-2(-1)\\ &= 2+2+3i\\ &= 4+3i\end{aligned}$$

Before negative numbers were understood at all, subtraction still existed. But there was a time when an expression like $2-6$ was meaningless (literally: people didn't know what to do with the idea and would avoid writing it). Negative numbers came along and *extended* non-negative numbers to make the number system more general and capable.

Similarly, imaginary numbers *extend* the non-imaginary numbers and end up being useful and powerful in their own right.

Problems

1. Simplify the following into $a + bi$ form and draw them in the complex plane:

   a. $\sqrt{-9} + 2$
   b. $(2 + i) - (1 + 3i)$
   c. $\sqrt{-4}(3 + 4i)$
   d. $(1 + 2i)^2$
   e. $(1 + 2i)^3$

Complex numbers as rotation

As you might have started to notice in the last two problems, multiplying by the same complex number repeatedly starts to turn the result counterclockwise in the complex plane. Let's look at this a little more closely.

Consider what happens when you multiply a number repeatedly by i:

$1 * i = i$

$1 * i * i = -1$

$1 * i * i * i = -i$

$1 * i * i * i * i = 1$

If you plot each number 1, i, -1, and $-i$ in the complex plane, you'll see something interesting: multiplying by i *rotates* by 90 degrees: 1 points to the right along the x/real axis, i points straight up along the imaginary axis, -1 points left, and $-i$ points straight down. And this isn't unique to starting with the number 1; any complex number is rotated the same way (see problems 2 and 3).

This suggests another useful way of describing a complex number. Instead of $a + bi$, we can describe a complex number's vector in the plane by two pieces of information: its length ($\sqrt{a^2 + b^2}$) and its angle (technically $\tan^{-1}$ of b/a but we won't have to use that at all here).

We can always get back to $a = bi$ form from an angle and a length: $a = l\cos(\theta)$ and $b = l\sin(\theta)$; or $l\cos(\theta) + il\sin(\theta)$. This is an important point that we'll come back to.

Let's describe a few numbers by their length and angle:

- i has length 1, and angle of 1/4 turn[1] (or 90 degrees).
- $2 + 2i$ has length $\sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$ and angle 1/8 turn (or 45 degrees).
- -3 has length 3, and angle 1/2 turn or 180 degrees.

Problems

2. Draw $(1 + 2i)$ in the complex plane. Find $(1 + 2i) * i$, and draw this in the imaginary plane. Do the same for $(1 + 2i) * i * i$, $(1 + 2i) * i * i * i$, etc. until you arrive back to $(1 + 2i)$.

[1] Here "turn" means full rotation around the circle; or 2π in radians.

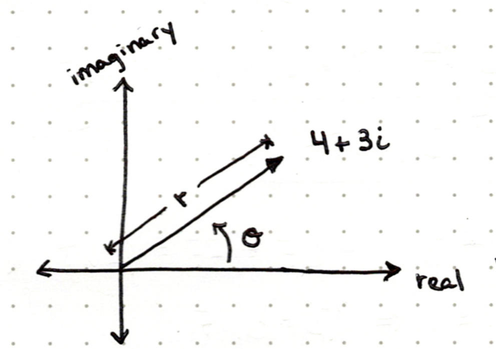

Figure 3: Here's the same complex number (4 + 3i) as we saw in Figure 2, but here we see its length r, and its angle θ. It's still in the same place in the complex plane, we're just measuring it differently.

3. Use the general $a+bi$ and multiply by i four times to show that any complex number times i^4 is rotated back to where it started.
4. Try multiplying complex numbers by $\frac{1}{\sqrt{2}}(1+i)$. What does this do?

(i.e. pick a number and draw it in the complex plane. multiply it by $\frac{1}{\sqrt{2}}(1+i)$ and draw it again - what happened to it? What happens if you multiply by $\frac{1}{\sqrt{2}}(1+i)$ again?)

5. What does multiplying complex numbers by $1+i$ do? Describe the effect this has on the direction and the length of the vector. (You may have to pick a number and multiply it by $1+i$ a few times to see the pattern.)

Euler's equation

You may have spotted this pattern too if we have two complex numbers n_1 and n_2 then their product is as long (in the complex plane) as their lengths *multiplied* together and has for its angle their angles *added* together.

As it turns out, this is a clue of how complex angles are related to exponentiation (where multiplying $x^a x^b = x^{a+b}$). We'll see that we can directly relate exponentiation, imaginary numbers and *rotation* in one equation.

Problems

6. Derive Euler's equation $e^{ix} = \cos(x) + i\sin(x)$.
   a) Recall or work out the Taylor series for e^x
   b) Replace x by (kx) to find the Taylor series for e^{kx} (k is a constant).
   c) Recall or work out the Taylor series for $\cos(x)$ and $\sin(x)$.
   d) Multiply the Taylor series for $\sin(x)$ by i to find the series for $i\sin(x)$

e) Find the Taylor series for e^{ix} by substituting $k = i$ and simplifying using $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, etc.

f) Finally, show that $e^{ix} = \cos(x) + i\sin(x)$.

Using Euler's equation

This is a remarkable result. It relates exponetiation to rotation, and allows us to write any complex number with length r and angle θ as $re^{i\theta}$; which makes the rule for multiplying complex numbers algebraically clear:

$$r_1 e^{i\theta_1} * r_2 e^{i\theta_2} = r_1 r_2 e^{i(theta_1 + \theta_2)}$$

It also leads directly to the famous:

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$$

and

$$e^{i2\pi} = \cos(2\pi) + i\sin(2\pi) = 1$$

One way to think about it is the following: Multiplying by i rotates by 90 degrees in the complex plane. e^x takes a multiplication by x and applies it smoothly and continuously (think of the compound interest definition of e). So if we combine these effects and consider $1 * e^{i2\pi}$, we're taking the vector 1 and smoothly rotating it 90 degrees from its current position in exactly such a way that it starts to draw a circle; and we're doing this until its traveled 2π distance – the circumference of a circle with radius 1.