Advanced Homework 5 - Wrapping Up

Due Date: June 1st

Here we are, at the final advanced homework! We'll keep it brief and just run through a review, a couple applications, and end with a feedback form.

Review

1. Find the Taylor series for:

   a. $\cos(x)$
   b. $\sin(x)$
   c. e^x
   d. e^{ix}

Notice how $e^{ix} = \cos(x) + i\sin(x)$ follows from the Taylor series you just derived.

2. Use Euler's equation ($e^{ix} = \cos(x) + i\sin(x)$) to find:

   a. $e^{i\pi}$
   b. $e^{i\pi/2}$
   c. $e^{i2\pi}$

Cosine and sine as exponentials

Use Euler's equation and the facts that $\cos(x)$ is even ($\cos(-x) = \cos(x)$) and $\sin(x)$ is odd ($\sin(-x) = -\sin(x)$) to show that:

3. $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$
4. $\sin(x) = \frac{1}{2}(e^{ix} - e^{-ix})$

Expand e^{ix} and e^{-ix} on the right side of the equation into cosines and sines, apply odd/evenness, and simplify.

Deriving trig identities

You can derive pretty much all the trigonometric identities using $e^{ix} = \cos(x) + i\sin(x)$[1].

For example, show:

[1] You'll ocassionally also need the pythagorean theorem: $\sin^2(x) + \cos^2(x) = 1$. Knowing how to derive all of these identities frees you from the need to ever memorize them, and can feel very satisfying and demystifying.

5. $\cos^2(x) = \frac{1}{2}\cos(2x) + \frac{1}{2}$
6. $\sin^2(x) = \frac{1}{2}\cos(2x) - \frac{1}{2}$

Squaring an exponential function multiplies its argument by two ($(e^x)^2 = e^{2x}$). This is not generally true of any other functions, but you can see how it's *kind of* true of $\cos(x)$ and $\sin(x)$ because they're essentially complex exponential functions.

Differential equations

$\frac{d^2}{dt^2}y = -y$ describes the motion of a pendulum swinging back and forth, or a mass on the end of a spring bouncing back and forth. The first time you see this in a physics class you learn that the solutions are $C_1 \sin(t)$ or $C_2 \cos(t)$.

7. Show that $\frac{d^2}{dt^2}y = -y$ is also solved by either $y = C_1 e^{it}$ or $y = C_2 e^{-it}$.

To do this, compare the second derivative of Ce^{it} to Ce^{it}; remember that $i^2 = -1$.

Complex exponentials show up in systems in nature that oscillate: pendulums, springs, waves on the water or in the air, and are most prominent in quantum mechanics in wave-particle duality.

Feedback on the advanced homework

8. Finally, please fill out a brief survey to let me know what you thought of the advanced homework! I've never tried to do anything like this before, but I appreciate you giving it a shot and I hope you found it rewarding and worth the effort.

https://forms.gle/kGZg26jotjqsRz19A